Geometric Algebra

Prof. Nigel Boston

Camera Network Research Group Meeting

Nov. 8 & 15, 2007

Strategy: Use Clifford algebra to develop invariants for projective transformations.

(Reference: J. Lasenby, et al., "New Geometric Methods for Computer Vision: an application to structure and motion estimation", 1996)

Idea: Basis-free geometry (since it is basis-free, invariants will naturally arise, therefore it should be useful for our purposes.)

Given: 2 vectors, **a**, **b**

$\mathbf{a} \cdot \mathbf{b}$ (grade 0)
$\mathbf{a} \times \mathbf{b}$ (grade 1) (only valid for 3D)

More generally, introduce

$\mathbf{a} \wedge \mathbf{b}$ (**a** 'wedge' **b**) is a directed area in sweeping from **a** to **b**.(see Fig. 1)
(a parallelogram) (this is a *bivector*)

And,

$$\mathbf{b} \wedge \mathbf{a} = - \mathbf{a} \wedge \mathbf{b} \quad (1)$$

In general, we have

$\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_{m-1} \wedge \mathbf{a}_m$ (*multi-vector*) (a parallelopiped) (grade m)

Define:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \quad (2)$$

it follows that:

$$\mathbf{ab} = \mathbf{a} \wedge \mathbf{b} \quad \text{if } \mathbf{a}, \mathbf{b} \text{ are orthogonal, and} \quad (3)$$

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} \quad \text{if } \mathbf{a}, \mathbf{b} \text{ are parallel} \quad (4)$$

$$\mathbf{ba} = \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \wedge \mathbf{a} = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \wedge \mathbf{b} \quad (5)$$

so then,

$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba})$ (associated *inner product*) (6)

$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba})$ (associated *outer product*) (7)

Definition:

A *geometric algebra* has a product satisfying:

$(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{bc})$ $\mathbf{a}\lambda = \lambda\mathbf{a}$, for scalar λ

$\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac}$ $\mathbf{a}^2 = |\mathbf{a}|^2$

$(\mathbf{b} + \mathbf{c})\mathbf{a} = \mathbf{ba} + \mathbf{ca}$

and the associated inner and outer product are defined by (6) and (7).

If $\{\sigma_1, \ldots, \sigma_n\}$ is any orthonormal basis for $\Re^n$, we get a basis for the algebra:

$\{1, \{\sigma_i\}, \{\sigma_i \wedge \sigma_j \mid i \neq j\}, \{\sigma_i \wedge \sigma_j \wedge \sigma_k \mid \#\{i, j, k\} = 3\}, \text{etc.}\}$

which will have 2^n elements. The highest grade element is $\sigma_i \wedge \ldots \wedge \sigma_n$.

<u>Example:</u> n = 3 The basis will have $2^n = 2^3 = 8$ elements:

$\{1, \sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_1\sigma_3, \sigma_1\sigma_2\sigma_3\}$

(Note: $\sigma_1 \cdot \sigma_2 = 0$ since the basis is orthonormal, hence $\sigma_1\sigma_2 = \sigma_1 \wedge \sigma_2$.)
(Also, since the basis is orthonormal, $\sigma_i^2 = |\sigma_i|^2 = 1$.)
(This is $\Re^8$ with a strange multiplication.)

It follows that:

$(\sigma_1\sigma_2\sigma_3)^2 = \sigma_1\sigma_2\sigma_3\,\sigma_1\sigma_2\sigma_3 = -\sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_3 = -\sigma_1^2\,\sigma_2\,\sigma_2\sigma_3^2 = -\sigma_1^2\,\sigma_2^2\,\sigma_3^2 = -1$

Also, one can verify that $\sigma_1\sigma_2\sigma_3$ commutes with every element of the algebra (will be true for n odd).

Call $\sigma_1\sigma_2\sigma_3 = i$ (i.e., $\sqrt{-1}$)

It follows that:

$i\,\sigma_3 = \sigma_1\sigma_2$, $i\,\sigma_1 = \sigma_2\sigma_3$, $i\,\sigma_2 = \sigma_3\sigma_1$ (8)

These bivectors rotate vectors in their own plane by 90°.

Note: set $I = i\,\sigma_1$, $J = -\,i\,\sigma_2$, $K = i\,\sigma_3$, then we get:
$I^2 = J^2 = K^2 = IJK = -1$ (Hamilton's quaternions).

The algebra with basis 1, I, J, K is $\Re^4$ with a strange multiplication, i.e., a sub-algebra of our geometric algebra.

General Rotations:

First, study *reflections*. (easier) (Any rotation is the product of two reflections.)

Consider a vector, **a** reflected in the plane orthogonal to the unit vector **n**, to get **a**′ (see Fig. 2).

Then, we can breakup **a** into components,

$$\mathbf{a} = \mathbf{a}_{\perp} + \mathbf{a}_{\parallel}, \qquad (9)$$

the components perpendicular and parallel to **n**, and then,

$$\mathbf{a}' = \mathbf{a}_{\perp} - \mathbf{a}_{\parallel}. \qquad (10)$$

Since **n** is a unit vector, then $\mathbf{n}^2 = |\mathbf{n}|^2 = 1$, and

$\mathbf{a} = \mathbf{n}^2\mathbf{a} = \mathbf{n}(\mathbf{n} \cdot \mathbf{a} + \mathbf{n} \wedge \mathbf{a}) = (\mathbf{n} \cdot \mathbf{a})\mathbf{n} + \mathbf{n}(\mathbf{n} \wedge \mathbf{a})$, which implies that

$$\begin{aligned} \mathbf{a}' = \mathbf{a}_{\perp} - \mathbf{a}_{\parallel} &= \mathbf{n}(\mathbf{n} \wedge \mathbf{a}) - (\mathbf{n} \cdot \mathbf{a})\mathbf{n} \\ &= -(\mathbf{n} \cdot \mathbf{a})\mathbf{n} - (\mathbf{n} \wedge \mathbf{a})\mathbf{n} \\ &= -(\mathbf{n} \cdot \mathbf{a} + \mathbf{n} \wedge \mathbf{a})\mathbf{n} = -\mathbf{nan}. \qquad (11) \end{aligned}$$

Thus, reflecting in the plane perpendicular to **n** is the map $\mathbf{a} \mapsto -\mathbf{nan}$
(Note: this works in any dimension)

Suppose you reflect in the plane perpendicular to **n**, and then in the plane perpendicular to **m**. (the product of two reflections) This is a rotation. Then, **a** maps to

$$\begin{aligned} -\mathbf{m}(-\mathbf{nan})\mathbf{m} &= (\mathbf{mn})\mathbf{a}(\mathbf{nm}) \\ &= R\mathbf{a}\tilde{R}, \quad \text{where } R = \mathbf{mn},\ \tilde{R} = \mathbf{nm}. \qquad (12) \end{aligned}$$

R is called a *rotor* (it is also a *multivector*) (it encapsulates the information about the rotation.)

Notes:

1) *R* only has even grade elements (scalar, bivector, etc.) and $R\tilde{R} = \mathbf{mnnm} = \mathbf{mn}^2\mathbf{m} = \mathbf{mm} = 1$, and $\tilde{R}R = 1$ ($\tilde{R}$ is the multiplicative inverse of R)
2) $\mathbf{a} \mapsto R\mathbf{a}\tilde{R}$ handles rotations in any dimension.
3) Can rotate elements of any grade, not just vectors – very general!

Consider the problem of rotating $\mathbf{n}_1$ to $\mathbf{n}_2$, say by angle θ (see Fig. 3). What is R?

$$\mathbf{n}_2 = R\mathbf{n}_1\tilde{R} \Rightarrow \mathbf{n}_2 R = R\mathbf{n}_1$$

Note that one solution is $R = 1 + \mathbf{n}_2\mathbf{n}_1$, but we also need $\tilde{R}R = 1$, so try $R = \alpha(1 + \mathbf{n}_2\mathbf{n}_1)$, then

$$1 = R\tilde{R} = \alpha^2(1+\mathbf{n}_2\mathbf{n}_1)(1+\mathbf{n}_1\mathbf{n}_2) = \alpha^2(1+1+\mathbf{n}_2\mathbf{n}_1+\mathbf{n}_1\mathbf{n}_2)$$

$$= \alpha^2(2+2\mathbf{n}_2\cdot\mathbf{n}_1) = 2\alpha^2(1+\mathbf{n}_2\cdot\mathbf{n}_1)$$

So, $$R = \frac{1+\mathbf{n}_2\mathbf{n}_1}{\sqrt{2(1+\mathbf{n}_2\cdot\mathbf{n}_1)}} = \exp\left(-i\frac{\theta}{2}\mathbf{n}\right) , \qquad (13)$$

where $\mathbf{n}$ is orthogonal to the plane cut out by $\mathbf{n}_1$ and $\mathbf{n}_2$ (the axis of rotation).

11/15/07

Example: Camera motion from two scene projections with range data known. (3D-to-3D correspondences)

Assume cameras with optical centers at O_1 and O_2, with respective Axes $\{\sigma_1, \sigma_2, \sigma_3\}$ and $\{\sigma_1', \sigma_2', \sigma_3'\}$, where σ_3 is orthogonal to α_1, the image plane for camera one, and similarly for camera two. (See Fig. 4.) Let

$$\mathbf{X} = \overrightarrow{O_1P}, \ \mathbf{x} = \overrightarrow{O_1M_1}, \ \mathbf{t} = \overrightarrow{O_1O_2} \qquad (14)$$

The frame $\{\sigma_1, \sigma_2, \sigma_3\}$ is rotated to a frame $\{\sigma_1', \sigma_2', \sigma_3'\}$ at O_2, where for R being the corresponding rotor, we have

$$\sigma_i' = R\sigma_i\tilde{R} \Rightarrow \sigma_i = R^{-1}\sigma_i' R \qquad (15)$$

Let $\breve{\mathbf{X}} = \overrightarrow{O_2P}$

Then,

$$\breve{\mathbf{X}} = \mathbf{X} - \mathbf{t} \qquad (16)$$

The observables (measurements) in the $\{\sigma_i\}$ and $\{\sigma_i'\}$ frames are:

$$X_i = \mathbf{X} \cdot \sigma_i$$

and

$$X_i' = \breve{\mathbf{X}} \cdot \sigma_i'$$

Define the vector $\mathbf{X}'$ in the $\{\sigma_i\}$frame as:

$$\mathbf{X}' = X_i'\sigma_i = X_i'\left(R^{-1}\sigma_i'R\right) = R^{-1}\left(X_i'\sigma_i'\right)R = R^{-1}\breve{\mathbf{X}}R = R^{-1}\left(\mathbf{X} - \mathbf{t}\right)R \quad (17)$$

Rearranging:

$$\mathbf{X} - \mathbf{t} = R\mathbf{X}'R^{-1} \Rightarrow \mathbf{X} = R\mathbf{X}'R^{-1} + \mathbf{t} \quad (18)$$

For simplicity, assume $|\mathbf{t}| = 1$.

Suppose we have n point correspondences in the two views, and the coordinates in the views, $\{\mathbf{X}_i\}$ and $\{\mathbf{X}_i'\}$, $(1 \le i \le n)$, are known.

We want to recover the camera motion, i.e., find R and $\mathbf{t}$ that minimize the sum:

$$S = \sum_{i=1}^{n} \left[\mathbf{X}_i' - R^{-1}\left(\mathbf{X}_i - \mathbf{t}\right)R\right]^2 \quad (19)$$

To find this minimization, we differentiate wrt R and $\mathbf{t}$, then set to zero. First, wrt $\mathbf{t}$:

$$\partial_t S = 2\sum_{i=1}^{n} \left[\mathbf{X}_i' - R^{-1}\left(\mathbf{X}_i - \mathbf{t}\right)R\right]\partial_t\left(R^{-1}\mathbf{t}R\right) \quad (20)$$

which will be zero when,

$$\sum_{i=1}^{n} \left[\mathbf{X}_i' - R^{-1}\left(\mathbf{X}_i - \mathbf{t}\right)R\right] = 0$$

Solve for $\mathbf{t}$:

$$\sum_{i=1}^{n} \left[\mathbf{X}_i' - R^{-1}\mathbf{X}_i R\right] + nR^{-1}\mathbf{t}R = 0 \quad \Leftrightarrow \quad R^{-1}\mathbf{t}R = \frac{1}{n}\sum_{i=1}^{n}\left[R^{-1}\mathbf{X}_i R - \mathbf{X}_i'\right]$$

$$\Leftrightarrow \quad \mathbf{t} = \frac{1}{n}\sum_{i=1}^{n}\left[\mathbf{X}_i - R\mathbf{X}_i'R^{-1}\right] = \overline{\mathbf{X}} - R\overline{\mathbf{X}}'R^{-1} \quad (21)$$

where

$$\overline{\mathbf{X}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i \quad \text{and} \quad \overline{\mathbf{X}}' = \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}'_i \tag{22}$$

So, the optimal **t** is the centroids of the points.
(Note: this is a known result that we have recovered here.)
(Note: there will be an issue with robustness since one outlier can adversely affect the results – a so-called 'black swan')

Then, differentiating wrt *R*, it can be shown that we get:

$$\sum_{i=1}^{n}\left[\mathbf{X}'_i \wedge R^{-1}(\mathbf{X}_i - \mathbf{t})R\right] = 0 \tag{23}$$

Plug-in the optimal **t** and we get:

$$\sum_{i=1}^{n}\left[\mathbf{v}_i \wedge R^{-1}\mathbf{u}_i R\right] = 0 \tag{24}$$

where $\mathbf{u}_i = \mathbf{X}_i - \overline{\mathbf{X}}$ and $\mathbf{v}_i = \mathbf{X}'_i$

We can find the rotor, *R*, using the Singular Value Decomposition (SVD) on the matrix **F**, defined in terms of the $\mathbf{u}_i$'s and $\mathbf{v}_i$'s as:

$$\mathbf{F}_{\alpha\beta} \equiv \sigma_\alpha \cdot \underline{f}(\sigma_\beta) = \sum_{i=1}^{n}(\sigma_\alpha \cdot \mathbf{u}_i)(\sigma_\beta \cdot \mathbf{v}_i) \tag{25}$$

then the SVD gives $\mathbf{F} = USV^{\mathrm{T}}$, and then $R = VU^{\mathrm{T}}$.

Paper for next time (Matt):

"A geometric approach for the theory and applications of 3D projective invariants"
Bayro-Corrochano, Eduardo; Banarer, Vladimir
Journal of Mathematical Imaging and Vision, v. 16, n. 2, March 2002, pp. 131-154

Appendix:

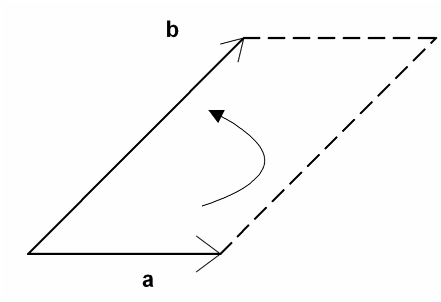

Fig. 1 - **a** /\ **b**

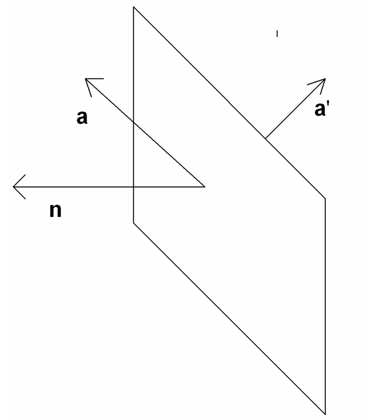

Fig. 2 – Vector **a** reflected in the plane orthogonal to **n.**

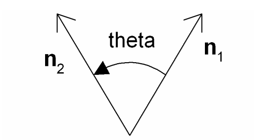

Fig. 3 – Rotation of $\mathbf{n}_1$ to $\mathbf{n}_2$.

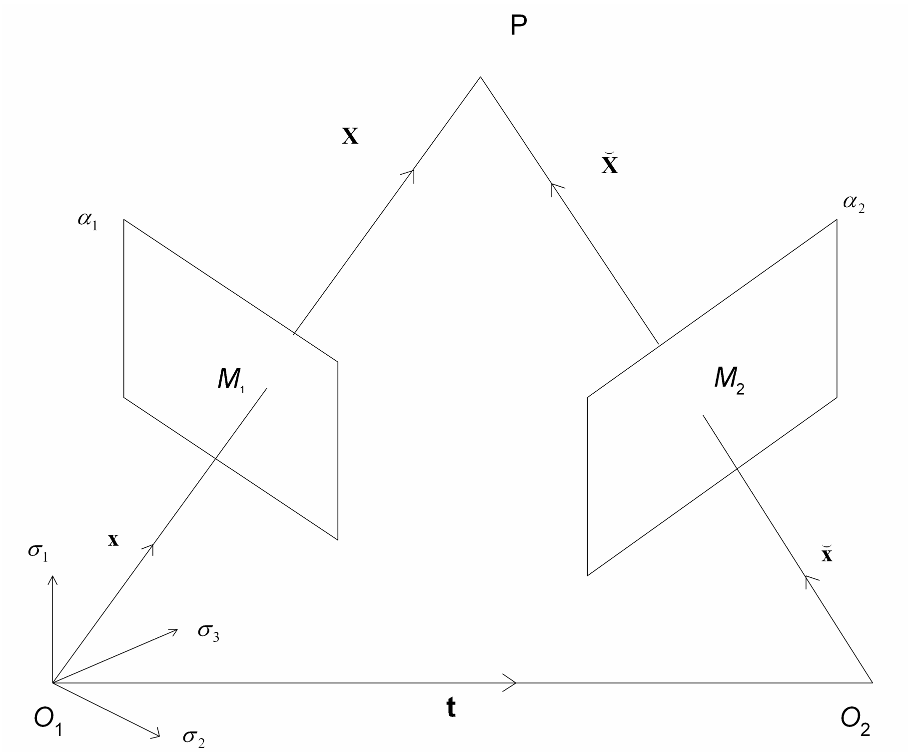

Fig. 4 – Object point viewed from two camera positions.